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# Homogenization of Dirichlet parabolic problems for coefficients and open sets simultaneously variable and applications to optimal design

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## Abstract

In a previous paper, we studied the homogenization of a sequence of parabolic linear Dirichlet problems, when the coefficients and the domains vary arbitrarily. Here, we improve the convergence result given in this paper by showing the strong convergence in  $L^2$  every time. This is applied to obtain an existence result for control problems in optimal design written in a relaxed form. The control variables are the material and the shape.

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## 1. Introduction

We are interested in the asymptotic behavior of a sequence of parabolic Dirichlet problems when the coefficients and the open sets where they are posed simultaneously vary. Specifically, for  $T > 0$ ,  $\Omega \subset \mathbb{R}^N$ , open,  $A_n : \Omega \times (0, T) \rightarrow \mathbb{R}^{N \times N}$ , elliptic and bounded,  $\Omega_n \subset \Omega$  open, and  $f \in L^2(0, T; H^{-1}(\Omega))$ , let us consider the homogenization problem

$$\partial_t y_n - \operatorname{div} A_n(x, t) \nabla y_n = f \quad \text{in } \Omega \times (0, T),$$

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$$y = 0 \quad \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times (0, T)). \quad (1.1)$$

We do not introduce any hypotheses about  $\Omega_n$  (only the fact that they are all contained in  $\Omega$ ). For  $A_n$ , we only assume it to be uniformly elliptic, and bounded. As it is usual in the homogenization of Dirichlet problems in varying domains (see, e.g., [5,6,9–18,26,27]), it is proved in [9] that the limit problem of (1.1) does not have the same structure. In the place of an equation such as

$$\partial_t y - \operatorname{div} A(x, t) \nabla y = f \quad \text{in } \Omega \times (0, T),$$

we find a bounded and elliptic matrix  $A$ , a nonnegative measure  $\mu$  and a positive and bounded  $\mu$ -measurable function  $F$ , such that the limit equation is

$$\partial_t y - \operatorname{div} A(x, t) \nabla y + F(x, t) y \mu = f \quad \text{in } \Omega \times (0, T). \quad (1.2)$$

The measure  $\mu$  vanishes on the sets of capacity zero, and then the functions in  $H_0^1(\Omega)$  have a representative which is well defined for it. However, it is not in general in  $H^{-1}(\Omega)$ , and not even a Radon measure. So, Eq. (1.2) does not hold in general in the sense of the distributions. Thus, we will prefer to write it in a variational form better than as a partial differential equation.

The above result is closely related to the fact that a control problem like

$$\min_{\tilde{\Omega} \subset \Omega \text{ open}} \int_{\Omega} |y - y_d|^2 dx \quad \begin{cases} \partial_t y - \Delta y = f & \text{in } \tilde{\Omega} \times (0, T), \\ y = 0 & \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times (0, T)), \end{cases}$$

with  $y_d$  in  $L^2(\Omega)$ , and  $f$  in  $L^2(0, T; H^{-1}(\Omega))$ , does not have a solution in general.

At the place of (1.1), we will prefer to consider the problem

$$\begin{aligned} y_n &\in L^2(0, T; H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega)), \quad y_n(x, 0) = 0 \text{ a.e. in } \Omega, \\ \langle \partial_t y_n, v \rangle + \int_{\Omega} A_n(x, t) \nabla y_n \nabla v dx + \int_{\Omega} F_n(x, t) y_n v d\mu_n &= \langle f, v \rangle \quad \text{in } \mathcal{D}'(0, T), \\ \forall v &\in L^2(0, T; H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega)), \end{aligned} \quad (1.3)$$

where  $A_n$  and  $f$  are as in (1.1),  $\mu_n$  is a sequence of nonnegative Borel measures which vanish on the sets of capacity zero, and  $F_n$  are in  $L_{\mu_n}^\infty(\Omega)$ , uniformly positive, and bounded. Following Dal Maso and Mosco [16], we remark that if  $\Omega_n$  is a sequence of open sets contained in  $\Omega$ , then, defining  $\mu_n$  as

$$\mu_n(B) = \begin{cases} +\infty & \text{if } \operatorname{Cap}(B \cap (\Omega \setminus \Omega_n), \Omega) > 0, \\ 0 & \text{if } \operatorname{Cap}(B \cap (\Omega \setminus \Omega_n), \Omega) = 0, \end{cases} \quad \forall B \subset \Omega \text{ Borel},$$

and, e.g.,  $F_n = \chi_{\Omega_n}$ , problem (1.1) is equivalent to (1.3), and so (1.3) generalizes (1.1).

The homogenization problem (1.1) has been studied in [9] (see also [5,17], for elliptic problems, and [6] for nonlinear parabolic problems where the coefficients do not depend on the time), where the existence of a limit problem is proved (for a subsequence), which has the same structure as (1.3). The convergence of  $y_n$  is proved to hold strong in  $L^2(\Omega \times (0, T))$  and weak in  $L^2(0, T; H_0^1(\Omega_n))$ . In the present paper, let us also show that for every  $t \in [0, T]$ ,  $y_n(\cdot, t)$  converges strongly in  $L^2(\Omega)$ . As an application of these results, we prove the existence of solutions for control problems in the coefficients and the domains. These problems must be written in a relaxed form. In other cases, it is well known that a solution does not exist in general (see, e.g., [3,7,22]). We refer to [1,3,4,7,8,21,22,24] for the study of control problems in optimal design.

## 2. Notations

We denote by  $\Omega \subset \mathbb{R}^N$  a bounded open set, by  $Q_R$ ,  $R > 0$ , the cylinder  $Q_R = \Omega \times (0, R)$ , and by  $Q_R^S$ ,  $0 < R < S$ , the cylinder  $Q_R = \Omega \times (R, S)$ .

For a measure  $\hat{\mu}$  in  $Q_R$ , we denote by  $L_{\hat{\mu}}^p(Q_R)$ ,  $1 \leq p \leq +\infty$ , the usual Lebesgue spaces relative to  $\hat{\mu}$ . If  $\hat{\mu}$  is the Lebesgue measure, we write  $L^p(Q_R)$ . Analogously, for a measure  $\mu$  in  $\Omega$ , we use the notations  $L_{\mu}^p\Omega$ ,  $L^p(\Omega)$ .

For a normed space  $X$ ,  $x \in X$ ,  $x' \in X'$  (the dual space of  $X$ ), we denote by  $\langle x', x \rangle_{x', x}$  the duality product between  $x'$  and  $x$ . When the spaces are understood, we just write  $\langle x', x \rangle$ .

For every  $B \subset \Omega$ ,  $\text{Cap}(B, \Omega)$  denotes the capacity of  $B$  (in  $\Omega$ ), which is defined as the infimum of

$$\int_{\Omega} |\nabla u|^2 dx$$

over the set of  $u \in H_0^1(\Omega)$  such that  $u \geq 1$  a.e. in a neighborhood of  $B$ .

A function  $u : \Omega \rightarrow \mathbb{R}$  is said to be quasi-continuous if for every  $\varepsilon > 0$  there exists  $N \subset \Omega$ , with  $C(N, \Omega) < \varepsilon$ , such that the restriction of  $u$  to  $\Omega \setminus N$  is continuous. It is well known that every function  $u \in H_0^1(\Omega)$  has a quasi-continuous representative (see [19,20,30]). We always identify  $u$  with its quasi-continuous representative.

A set  $\Theta \subset \Omega$  is said to be quasi-open, if for every  $\varepsilon > 0$  there exists  $N$  with  $C(N, \Omega) < \varepsilon$  such that  $\Theta \cup N$  is open.

We denote by  $\mathcal{M}_0^2(\Omega)$  the class of all nonnegative Borel measures which vanish on the sets of capacity zero and satisfy

$$\mu(B) = \inf\{\mu(\Theta) : \Theta \text{ quasi-open, } B \subseteq \Theta \subseteq \Omega\}, \quad \forall B \subset \Omega \text{ Borel.}$$

For a measure  $\mu \in \mathcal{M}_0^2(\Omega)$ , we denote by  $\hat{\mu}$  the measure in  $Q_T$  defined by  $\hat{\mu} = \mu \otimes dt$ .

**Definition 2.1.** For  $T > 0$ , and two constants  $\gamma > \alpha > 0$ , we denote by  $M_{\alpha}^{\gamma}(Q_T)$  (see [23]) the set of all the matrices  $A$  in  $L^{\infty}(Q_T)^{N \times N}$ , such that

- (i)  $A(x, t)\xi\xi \geq \alpha|\xi|^2$ ,  $\forall \xi \in \mathbb{R}^N$ , a.e.  $(x, t) \in Q_T$ .
- (ii)  $A^{-1}(x, t)\xi\xi \geq \gamma^{-1}|\xi|^2$ ,  $\forall \xi \in \mathbb{R}^N$ , a.e.  $(x, t) \in Q_T$ .

We also denote by  $\mathcal{F}_{\alpha}^{\gamma}(Q_T)$  the set of pairs  $(F, \mu)$  such that  $\mu \in \mathcal{M}_0^2(\Omega)$ ,  $F$  belongs to  $L_{\hat{\mu}}^{\infty}(Q_T)$ , and

$$\gamma \geq F(x, t) \geq \alpha, \quad \hat{\mu}\text{-a.e. in } Q_T. \quad (2.4)$$

**Remark 2.2.** We recall (see [23]) that (ii) implies

- (iii)  $|A(x, t)| \leq \gamma$ , a.e.  $(x, t) \in Q_T$ .

Reciprocally, if  $A$  satisfies (i) and (iii), then

$$A^{-1}(x, t)\xi\xi \geq \frac{\alpha}{\gamma^2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. } (x, t) \in Q_T.$$

### 3. Homogenization results

We recall in this section the following compactness result, which gives the homogenization of (1.3) (see also [5,18], for the case of elliptic equations, and [6] for the case of nonlinear parabolic problems with coefficients independent of the time variable.

**Theorem 3.1.** *For  $T > 0$ ,  $\gamma > \alpha > 0$ , and two sequences  $A_n \in \mathcal{M}_\alpha^\gamma(Q_T)$  and  $(F_n, \mu_n) \in \mathcal{F}_\alpha^\gamma(Q_T)$ , there exist a subsequence of  $n$ , still denoted by  $n$ ,  $A \in \mathcal{M}_\alpha^\gamma(Q_T)$  and  $(F, \mu) \in \mathcal{F}_\alpha^\gamma(Q_T)$ , such that for every distribution  $f \in L^2(0, T; H^{-1}(\Omega))$ , the solution  $y_n$  of (1.3) converges weakly in  $L^2(0, T; H_0^1(\Omega))$  and strongly in  $L^2(Q_T)$  to the unique solution  $y$  of*

$$\begin{aligned} y &\in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)), \quad y(x, 0) = 0 \text{ a.e. in } \Omega, \\ \langle \partial_t y, v \rangle + \int_\Omega A(x, t) \nabla y \nabla v \, dx + \int_\Omega F(x, t) y v \, d\mu &= \langle f, v \rangle \quad \text{in } \mathcal{D}'(0, T), \\ \forall v &\in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)). \end{aligned} \quad (3.5)$$

The matrix  $A$  coincides with the  $H$ -limit of  $A_n$  (see, e.g., [23,25,28]), and then, it does not depend on  $(F_n, \mu_n)$ . The measure  $\mu$  can be chosen (note that only the product  $F\mu$  is uniquely defined) as the unique element of  $\mathcal{M}_0^2(\Omega)$  (see [15]), such that the unique solution  $w_n$  of

$$\begin{aligned} w_n &\in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega), \\ \int_\Omega \nabla w_n \nabla v \, dx + \int_\Omega w_n v \, d\mu_n &= \int_\Omega w_n v \, dx, \\ \forall v &\in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega) \end{aligned}$$

converges weakly in  $H_0^1(\Omega)$  to the unique solution  $w$  of

$$\begin{aligned} w &\in H_0^1(\Omega) \cap L_\mu^2(\Omega), \\ \int_\Omega \nabla w \nabla v \, dx + \int_\Omega w v \, d\mu &= \int_\Omega w v \, dx, \\ \forall v &\in H_0^1(\Omega) \cap L_\mu^2(\Omega), \end{aligned} \quad (3.6)$$

and then, it can be chosen independently of  $A_n$  and  $F_n$ .

Let us improve the above result by showing the following:

**Proposition 3.2.** *In Theorem (3.1), we also have*

$$y_n(., t) \rightarrow y(., t) \quad \text{in } L^2(\Omega), \quad \forall t \in [0, T]. \quad (3.7)$$

**Proof.** Let  $t$  be in  $[0, T]$ ; there is nothing to prove  $t = 0$ . So, we can assume  $t \in (0, T]$ . Moreover, it is not restrictive to assume that  $y_n$  and  $y$  are defined in  $Q_S$  for some  $S > T$ , and that Theorem 3.1 holds with  $T$  replaced by  $S$ . For this, it will be enough to extend  $A_n$ , and  $F_n$  to  $Q_S$ .

For  $\varepsilon > 0$ , we consider  $h \in (0, \min\{t/2, (S-t)/2\})$  such that

$$\|\nabla y\|_{L^2(Q_{t-2h}^{t+2h})} + \|y\|_{L^2_{\mu}(Q_{t-2h}^{t+2h})} + \frac{\alpha}{\gamma^2} \|f\|_{L^2(t-h, t+h; H^{-1}(\Omega))} < \varepsilon. \quad (3.8)$$

Since the solutions  $y_n$  of (1.3) are in  $C^0([0, S]; L^2(\Omega))$ , for every  $n \in N$ , there exists  $h_n \in (0, h)$  such that

$$\left\| y_n(\cdot, t) - \frac{1}{2h_n} \int_{t-h_n}^{t+h_n} y_n(\cdot, s) \right\|_{L^2(\Omega)} < \varepsilon. \quad (3.9)$$

Using (1.3), for every  $n \in N$ , and a.e.  $(r, s) \in (t-h, t+h)^2$ , we have

$$\begin{aligned} & \left\langle \frac{\partial y_n}{\partial r}(x, r), y_n(\cdot, r) - y_n(\cdot, s) \right\rangle_{(H_0^1(\Omega) \cap L^2_{\mu_n}(\Omega))', H_0^1(\Omega) \cap L^2_{\mu_n}(\Omega)} \\ & + \int_{\Omega} A_n(x, r) \nabla y_n(x, r) \nabla (y_n(x, r) - y_n(x, s)) \, dx \\ & + \int_{\Omega} F_n(x, r) y_n(x, r) (y_n(x, r) - y_n(x, s)) \, d\mu_n \\ & = \langle f, y_n(\cdot, r) - y_n(\cdot, s) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned}$$

Integrating in  $r \in (q, s)$ , for  $q \in (t-h, s)$ , or in  $r \in (s, q)$ , for  $q \in (s, t+h)$ , we get

$$\begin{aligned} \int_{\Omega} |y_n(x, q) - y_n(x, s)|^2 \, dx & \leq \gamma \|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} \|\nabla (y_n - y_n(\cdot, s))\|_{L^2(Q_{t-h}^{t+h})} \\ & + \gamma \|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})} \|y_n - y_n(\cdot, s)\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})} \\ & + \|f\|_{L^2(t-h, t+h; H^{-1}(\Omega))} \|\nabla (y_n - y_n(\cdot, s))\|_{L^2(Q_{t-h}^{t+h})}, \end{aligned}$$

for a.e.  $(q, s) \in (t-h, t+h)^2$ . Integrating in  $(q, s) \in (t-h_n, t+h_n) \times (t-h, t+h)$ , and dividing by  $4h_n h$  we obtain

$$\begin{aligned} & \frac{1}{4h_n h} \int_{t-h_n}^{t+h_n} \int_{t-h}^{t+h} \int_{\Omega} |y_n(x, q) - y_n(x, s)|^2 \, dx \, ds \, dq \\ & \leq (\gamma \|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \|f\|_{L^2(t-h, t+h; H^{-1}(\Omega))}) \\ & \quad \times \left( \frac{1}{2h} \int_{t-h}^{t+h} \int_{t-h}^{t+h} \int_{\Omega} |\nabla (y_n(x, s) - y_n(x, r))|^2 \, dx \, ds \, dr \right)^{1/2} \\ & + \gamma \|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})} \left( \frac{1}{2h} \int_{t-h}^{t+h} \int_{t-h}^{t+h} \int_{\Omega} |y_n(x, s) - y_n(x, r)|^2 \, d\mu_n \, ds \, dr \right)^{1/2} \\ & \leq \sqrt{2} (\gamma \|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \gamma \|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})} + \|f\|_{L^2(t-h, t+h; H^{-1}(\Omega))}) \\ & \quad \times (\|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})}). \end{aligned} \quad (3.10)$$

Now, for  $\varphi \in \mathcal{D}(t-2h, t+2h)$ ,  $\varphi \geq 0$ ,  $\varphi = 1$  in  $(t-h, t+h)$ , we take the application  $(x, t) \rightarrow y_n(x, t)\varphi(t)$  as test function in (1.3), and the application  $(x, t) \rightarrow y(x, t)\varphi(t)$  as test function in (3.5). Using then that  $y_n$  converges to  $y$  strongly in  $L^2(Q_S)$  and weakly in  $L^2(0, S; H^{-1}(\Omega))$ , we have

$$\begin{aligned} \int_{Q_S} A_n \nabla y_n \nabla y_n \varphi \, dx \, dt + \int_{Q_S} F_n y_n^2 \varphi \, d\mu_n \, dt &= \frac{1}{2} \int_{\Omega} y_n^2 \frac{d\varphi}{dt} \, dx \, dt + \langle f, y_n \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\rightarrow \frac{1}{2} \int_{\Omega} y^2 \frac{d\varphi}{dt} \, dx \, dt + \langle f, y \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \int_{Q_S} A \nabla y \nabla y \varphi \, dx \, dt + \int_{Q_S} F y^2 \varphi \, d\mu \, dt. \end{aligned}$$

So, using the properties of  $A_n$ ,  $F_n$ ,  $A$ , and  $F$ , we get the following estimate to the right-hand side of (3.10):

$$\limsup_{n \rightarrow \infty} (\|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})}) \leq \frac{\gamma}{\alpha} (\|\nabla y\|_{L^2(Q_{t-h}^{t+h})} + \|y\|_{L^2_{\mu}(Q_{t-h}^{t+h})}). \quad (3.11)$$

Let us now consider the inequality

$$\begin{aligned} \|y_n(\cdot, t) - y(\cdot, t)\|_{L^2(\Omega)} &\leq \left\| y_n(\cdot, t) - \frac{1}{2h_n} \int_{t-h_n}^{t+h_n} y_n(\cdot, q) \, dq \right\|_{L^2(\Omega)} \\ &\quad + \left\| \frac{1}{4h_n h} \int_{t-h_n}^{t+h_n} \int_{t-h}^{t+h} (y_n(\cdot, q) - y_n(\cdot, s)) \, ds \, dq \right\|_{L^2(\Omega)} \\ &\quad + \left\| \frac{1}{2h} \int_{t-h}^{t+h} (y_n(\cdot, s) - y(\cdot, s)) \, ds \right\|_{L^2(\Omega)} \\ &\quad + \left\| \frac{1}{2h} \int_{t-h}^{t+h} y(\cdot, s) \, ds - y(\cdot, t) \right\|_{L^2(\Omega)}. \end{aligned}$$

From the strong convergence in  $L^2(Q_S)$  of  $y_n$  to  $y$ , the Cauchy–Schwartz inequality, (3.8)–(3.11), we can pass to the limit in this inequality to get

$$\limsup_{n \rightarrow \infty} \|y_n(\cdot, t) - y(\cdot, t)\|_{L^2(\Omega)} \leq \varepsilon + \frac{\sqrt{2}\gamma^3}{\alpha^2} \varepsilon^2 + \left\| \frac{1}{2h} \int_{t-h}^{t+h} y(\cdot, s) \, ds - y(\cdot, t) \right\|_{L^2(\Omega)}.$$

In this inequality  $h$  can be chosen as small as we want, since  $u$  belongs to  $C^0([0, S]; L^2(\Omega))$ . We can then pass to the limit when  $h$  tends to zero to obtain

$$\limsup_{n \rightarrow \infty} \|y_n(\cdot, t) - y(\cdot, t)\|_{L^2(\Omega)} \leq \varepsilon + \frac{\sqrt{2}\gamma^3}{\alpha^2} \varepsilon^2, \quad \forall \varepsilon > 0,$$

and then (3.7).  $\square$

#### 4. Existence of solution for optimal design problems

In this section, we investigate the existence of solution for the following control problem:

$$\min_{\tilde{\Omega} \in \mathcal{O}, A \in \mathcal{A}} J(y) \quad \begin{cases} \partial_t y - \operatorname{div} A(x, t) \nabla y = f & \text{in } \tilde{\Omega} \times (0, T), \\ y = 0 & \text{on } (\tilde{\Omega} \times \{0\}) \cup (\partial \tilde{\Omega} \times (0, T)), \end{cases} \quad (4.12)$$

where  $f$  belongs to  $L^2(0, T; H_0^1(\Omega))$ ,  $J$  is a functional in  $L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ ,  $\mathcal{O}$  is composed by open subsets of  $\Omega$ , and  $\mathcal{A}$  is a subset of  $\mathcal{M}_\alpha^\gamma(Q_T)$ . This type of problems arise in the optimization of materials (represented by the matrix  $A$ ) and shapes (represented by the open set  $\tilde{\Omega}$ ). It is well known that a problem like (4.12) has not a solution in general (see, e.g., [3,7,22]), and then, it is necessary to take a relaxation. In fact, because from Theorem 3.1, it is problem (3.5) which is stable by homogenization, it is better to replace (4.12) by

$$\min_{(A, (F, \mu)) \in \mathcal{E}} J(y) \quad \begin{cases} y \in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)), & y(x, 0) = 0, \text{ a.e. in } \Omega, \\ \langle \partial_t y, v \rangle + \int_\Omega A(x, t) \nabla y \nabla v \, dx \\ \quad + \int_\Omega F(x, t) y v \, d\mu = \langle f, v \rangle & \text{in } \mathcal{D}'(0, T), \\ \forall v \in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)), \end{cases} \quad (4.13)$$

with  $\mathcal{E}$  a subset of  $\mathcal{M}_\alpha^\gamma(Q_T) \times \mathcal{F}_\alpha^\gamma(Q_T)$ . Using the direct method of the calculus of variations, Theorem 3.1 and Proposition 3.2 can be immediately proved.

**Theorem 4.1.** *For  $T > 0$ ,  $\gamma > \alpha > 0$ , let  $\mathcal{E}$  be a subset of  $\mathcal{M}_\alpha^\gamma(Q_T) \times \mathcal{F}_\alpha^\gamma(Q_T)$  stable by homogenization, i.e., such that the limit of a sequence of problems like (1.3), with  $(A_n, (F_n, \mu_n)) \in \mathcal{E}$ , is of the form (3.5), with  $(A, (F, \mu)) \in \mathcal{E}$ , and let  $J : L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega)) \rightarrow \mathbb{R}$  be a functional which is semicontinuous in the following sense:*

*For every sequence  $y_n \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ , which is bounded in  $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ , and converges to  $y \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ , weakly in  $L^2(0, T; H_0^1(\Omega))$ , strongly in  $L^2(Q_T)$ , and also  $y_n(\cdot, t)$  converges strongly in  $L^2(\Omega)$  to  $y(\cdot, t)$ , for every  $t \in [0, T]$ , we have*

$$\liminf_{n \rightarrow \infty} J(y_n) \geq J(y).$$

*Then, for every  $f \in L^2(0, T; H^{-1}(\Omega))$ , problem (4.13) has at least a solution.*

As examples of functionals  $J$  in the conditions of Theorem 4.1, we have

$$y \rightarrow \int_\Omega |y(x, T) - y_d|^2 \, dx, \quad y_d \in L^2(\Omega),$$

$$y \rightarrow \int_{Q_T} |y(x, T) - y_d|^2 \, dx \, dt, \quad y_d \in L^2(Q_T),$$

$$y \rightarrow \int_{Q_T} |\nabla(y(x, T) - y_d)|^2 \, dx \, dt, \quad y_d \in L^2(0, T; H_0^1(\Omega)),$$

with respect to subsets  $\mathcal{E}$  in the conditions of Theorem 4.1. Thanks to Theorem 3.1, we can take  $\mathcal{E} = \mathcal{M}_\alpha^\gamma(Q_T) \times \mathcal{F}_\alpha^\gamma(Q_T)$ , but it is too large. In practice we only dispose of a few of materials and shapes.

Moreover, the question remains whether problem (4.13) is a relaxation of problem (4.12) or not. In this sense, the following definition is useful:

**Definition 4.2.** Given a subset  $\mathcal{E}$  of  $\mathcal{M}_\alpha^\gamma(Q_T) \times \mathcal{F}_\alpha^\gamma(Q_T)$ , we define the closure by homogenization of  $\mathcal{E}$ , and we denote it by  $C_H(\mathcal{E})$ , as the set of pairs  $(A, (F, \mu)) \in \mathcal{M}_\alpha^\gamma(Q_T) \times \mathcal{F}_\alpha^\gamma(Q_T)$ , such that there exists  $(A_n, (F_n, \mu_n)) \in \mathcal{E}$ , which satisfies that for every  $f \in L^2(0, T; H^{-1}(\Omega))$ , the unique solution of (1.3) converges weakly in  $L^2(0, T; H_0^1(\Omega))$  to the unique solution of (3.5).

From Theorem 3.1, it is clear that the closure by homogenization of a set  $\mathcal{E}$  is stable by homogenization, and then, it is in the conditions of Theorem 4.1. We easily prove the following:

**Proposition 4.3.** For  $T > 0$ ,  $\gamma > \alpha > 0$ , let  $\mathcal{E}$  be a subset of  $\mathcal{M}_\alpha^\gamma(Q_T) \times \mathcal{F}_\alpha^\gamma(Q_T)$ , and let  $J : L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega)) \rightarrow \mathbb{R}$  be a functional which satisfies the following continuity property:

For every sequence  $y_n \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ , which is bounded in  $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ , and converges to  $y \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ , weakly in  $L^2(0, T; H_0^1(\Omega))$ , strongly in  $L^2(Q_T)$ , and also  $y_n(\cdot, t)$  converges strongly in  $L^2(\Omega)$  to  $y(\cdot, t)$ , for every  $t \in [0, T]$ , we have

$$\lim_{n \rightarrow \infty} J(y_n) = J(y).$$

Then, for every distribution  $f \in L^2(0, T; H^{-1}(\Omega))$ , we get a relaxation of problem 4.13, just by replacing  $\mathcal{E}$  by  $C_H(\mathcal{E})$ .

**Remark 4.4.** The functional

$$y \rightarrow \int_{Q_T} |\nabla(y(x, T) - y_d)|^2 dx dt, \quad y_d \in L^2(0, T; H_0^1(\Omega))$$

satisfies the assumptions of Theorem 4.1 but not those of Proposition 4.3.

**Remark 4.5.** Since in Theorem 3.1  $A$  is the homogenized matrix of the sequence  $A_n$ , it is clear that for  $\mathcal{E} \subset \mathcal{M}_\alpha^\gamma(Q_T) \times \mathcal{F}_\alpha^\gamma(Q_T)$ , the projection of  $C_H(\mathcal{E})$  on  $\mathcal{M}_\alpha^\gamma(Q_T)$  coincides with the closure by  $H$ -convergence ( $H$ -closure) (see, e.g., [23,25,28]) of the projection of  $\mathcal{E}$  on  $\mathcal{M}_\alpha^\gamma(Q_T)$ .

From Proposition 4.3, in order to obtain a relaxation of (4.12), we need to obtain the closure by homogenization of the set of pairs  $\tilde{\Omega} \in \mathcal{O}$ ,  $A \in \mathcal{A}$ , where  $\mathcal{O}$  is composed of open subsets of  $\Omega$ , and  $\mathcal{A}$  is contained in  $\mathcal{M}_\alpha^\gamma(Q_T)$ . Here, we identify an open set  $\tilde{\Omega} \subset \Omega$ , with the pair  $(F, \mu) \in \mathcal{F}_\alpha^\gamma(Q_T)$ , given by

$$\mu(B) = \begin{cases} +\infty & \text{if } \text{Cap}(B \cap (\Omega \setminus \tilde{\Omega}), \Omega) > 0, \\ 0 & \text{if } \text{Cap}(B \cap (\Omega \setminus \tilde{\Omega}), \Omega) = 0, \end{cases} \quad \forall B \subset \Omega \text{ Borel},$$

and  $F = \frac{\alpha+\gamma}{2} \chi_{\tilde{\Omega}}$ .

When  $\mathcal{E}$  is of the form

$$\mathcal{E} = \mathcal{A} \times \{\tilde{\Omega} : \tilde{\Omega} \subset \Omega \text{ open}\},$$

with  $\mathcal{A}$  a subset of  $\mathcal{M}_\alpha^\gamma(Q_T)$  composed of constant matrices with respect to the time variable, we can use the results which appear in [2] to prove

$$C_H(\mathcal{E}) = \bar{\mathcal{A}} \times \{(F, \mu) \in \mathcal{F}_\alpha^\gamma(Q_T) : F(x, t) \text{ constant with respect to } t\},$$



with  $\bar{\mathcal{A}}$  the  $H$ -closure of  $\mathcal{A}$ . So, in this case the relaxation of problem (4.12) is reduced to the calculus of the  $H$ -closure of  $\mathcal{A}$  (which is only known to a very few choices of sets  $\mathcal{A}$ , see, e.g., [1,21,29]). Indeed, because for  $\mu \in \mathcal{M}_0^2(\Omega)$ , and  $F \in L_\mu^\infty(\Omega)$  constant with respect to the time variable, the product  $F\mu$  also gives a measure in  $\mu \in \mathcal{M}_0^2$ , for the above choice of  $\mathcal{E}$ , a relaxation of (4.13) is given by

$$\min_{(A,\mu) \in \bar{\mathcal{A}} \times \mathcal{M}_0^2(\Omega)} J(y) \quad \begin{cases} y \in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)), & y(x, 0) = 0 \text{ a.e. in } \Omega, \\ \langle \partial_t y, v \rangle + \int_\Omega A(x) \nabla y \nabla v \, dx + \int_\Omega y v \, d\mu = \langle f, v \rangle & \text{in } \mathcal{D}'(0, T), \\ \forall v \in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)). \end{cases}$$

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